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A complete convergence theorem for row sums from arrays of rowwise independent random elements in Rademacher type p Banach spaces. II

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ABSTRACT
In this correspondence, for an array of rowwise independent random elements $\{V_{n,k}, 1 \le k \le k_n, n \ge 1, k_n \to \infty\}$ taking values in a real separable Rademacher type p $(1 \le p \le 2)$ Banach space and a sequence of positive constants $\{c_n, n \ge 1\}$, the main result provides conditions for the complete convergence result $\sum_{n=1}^{\infty} c_n P\left(\max_{1\le k\le k_n}\left\|\sum_{i=1}^{k} V_{n,i}\right\| > \varepsilon\right) < \infty$ for all $\varepsilon > 0$ to hold. The complete convergence does not necessary hold if the Rademacher type p hypothesis is dispensed with. Corollaries of the main result are obtained and illustrative examples are presented.

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1. Introduction

This article is a follow-up to the article by Hu, Rosalsky, and Volodin [1] establishing complete convergence for row sums from arrays of rowwise independent random elements taking values in a real separable Rademacher type p $(1 \le p \le 2)$ Banach space. The reader may refer to Hu, Rosalsky, and Volodin [1] for a discussion regarding the relevant literature on complete convergence and for a review of the following technical definitions and concepts that pertain to the current work: complete convergence, Rademacher type p $(1 \le p \le 2)$ Banach space, random element V, $E(V)$, and rowwise independent. Throughout, all random elements are defined on a probability space $(\Omega, \mathcal{F}, P)$ and take values in a real separable Banach space $\mathcal{X}$ with norm $||\cdot||$ and $\{k_n, n \ge 1\}$ is a sequence of positive integers with $k_n \to \infty$ as $n \to \infty$.

At the origin of the current investigation is the following complete convergence theorem of Hu, Rosalsky, and Volodin [1].

Theorem 1.1. *(Hu, Rosalsky, and Volodin [1], Theorem 3.1). Let $\{V_{n,k}, 1 \le k \le k_n, n \ge 1\}$ be an array of rowwise independent random elements taking values in a real separable*

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Rademacher type p $(1 \leq p \leq 2)$ Banach space and let $\{c_n, n \geq 1\}$ be a sequence of positive constants. Suppose for some $J > 0$ and some $\delta_1, \delta_2 > 0$ that

$$\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P(||V_{n,k}|| > \varepsilon) < \infty \text{ for all } \varepsilon > 0, \tag{1.1}$$

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} E\big(||V_{n,k}||^p I(||V_{n,k}|| \leq \delta_1)\big) \right)^J < \infty, \tag{1.2}$$

and

$$\sum_{k=1}^{k_n} E(V_{n,k} I(||V_{n,k}|| \leq \delta_2)) \to 0 \text{ as } n \to \infty. \tag{1.3}$$

Then

$$\sum_{n=1}^{\infty} c_n P\left(\left\| \sum_{k=1}^{k_n} V_{n,k} \right\| > \varepsilon \right) < \infty \text{ for all } \varepsilon > 0. \tag{1.4}$$

Theorem 1.1 extends Theorem 1 of Sung, Volodin, and Hu [2] and Theorem 3.1 of Hu, Rosalsky, and Wang [3]. Theorem 3.1, the main result in the current work, is a complete convergence theorem and it relates to Theorem 1.1 as follows. A strengthening of condition (1.3) to (3.3) results in a strengthening of the conclusion (1.4) to (3.4). It is clear that Theorems 1.1 and 3.1 as well as the subsequent corollaries of Theorem 3.1 only have content if $\sum_{n=1}^{\infty} c_n = \infty$.

The plan of the paper is as follows. The lemmas employed for proving Theorem 3.1 are presented in Section 2. Theorem 3.1 is stated and proved in Section 3. Five corollaries of Theorem 3.1 are presented in Section 4 and four illustrative examples are presented in Section 5.

2. Lemmas

The following three lemmas are used to establish Theorem 3.1.

Lemma 2.1. (Ledoux and Talagrand [4], Proposition 6.7, p. 155). *Let $V_k, 1 \leq k \leq n$ be independent random elements taking values in a real separable Banach space. Then*

$$P\left(\max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} V_i \right\| > 3s + t \right) \leq P\Big(\max_{1 \leq k \leq n} ||V_k|| > t \Big) + \left(P\left(\max_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} V_i \right\| > s \right) \right)^2, \quad s > 0, t > 0.$$

Lemma 2.2. *For all integers $j \geq 0$, there exists a constant $0 < C_j < \infty$ depending only on j such that for all $n \geq 1$ and every set $\{V_k, 1 \leq k \leq n\}$ of n independent random elements taking values in a real separable Banach space,*

$$P\left(\max_{1\le k\le n}\left|\left|\sum_{i=1}^{k}V_i\right|\right|>(3^{j+1}-1)t\right)\le C_jP\Big(\max_{1\le k\le n}||V_k||>2t\Big)+\left(P\left(\max_{1\le k\le n}\left|\left|\sum_{i=1}^{k}V_i\right|\right|>2t\right)\right)^{2^j},\quad t>0. \tag{2.1}$$

Proof. For $j=0$, the inequality (2.1) is immediate with $C_0=1$. We proceed by induction on j. Suppose that (2.1) holds for some $j\ge 0$. Then for $j+1$, we have for all $t>0$ by Lemma 2.1 with s replaced by $(3^{j+1}-1)t$ and t replaced by $2t$ that

$$\begin{aligned}
&P\left(\max_{1\le k\le n}\left|\left|\sum_{i=1}^{k}V_i\right|\right|>(3^{j+2}-1)t\right)\\
&\quad=P\left(\max_{1\le k\le n}\left|\left|\sum_{i=1}^{k}V_i\right|\right|>3(3^{j+1}-1)t+2t\right)\\
&\quad\le P\Big(\max_{1\le k\le n}||V_k||>2t\Big)+\left(P\left(\max_{1\le k\le n}\left|\left|\sum_{i=1}^{k}V_i\right|\right|>(3^{j+1}-1)t\right)\right)^2\\
&\quad\le P\Big(\max_{1\le k\le n}||V_k||>2t\Big)+\left(C_jP\Big(\max_{1\le k\le n}||V_k||>2t\Big)+\left(P\left(\max_{1\le k\le n}\left|\left|\sum_{i=1}^{k}V_i\right|\right|>2t\right)\right)^{2^j}\right)^2\\
&\qquad\text{(by the induction hypothesis)}\\
&\quad=P\Big(\max_{1\le k\le n}||V_k||>2t\Big)+C_j^2\Big(P\Big(\max_{1\le k\le n}||V_k||>2t\Big)\Big)^2\\
&\qquad+2C_jP\Big(\max_{1\le k\le n}||V_k||>2t\Big)\left(P\left(\max_{1\le k\le n}\left|\left|\sum_{i=1}^{k}V_i\right|\right|>2t\right)\right)^{2^j}\\
&\qquad+\left(P\left(\max_{1\le k\le n}\left|\left|\sum_{i=1}^{k}V_i\right|\right|>2t\right)\right)^{2^{j+1}}\\
&\quad\le(1+C_j^2+2C_j)P\Big(\max_{1\le k\le n}||V_k||>2t\Big)+\left(P\left(\max_{1\le k\le n}\left|\left|\sum_{i=1}^{k}V_i\right|\right|>2t\right)\right)^{2^{j+1}}.
\end{aligned}$$

Hence taking $C_{j+1}=1+C_j^2+2C_j$ completes the proof. □

Lemma 2.3. (Rosalsky and Van Thanh [5], Lemma 2.1). *Let $\mathcal{X}$ be a real separable Rademacher type p $(1\le p\le 2)$ Banach space. Then there exists a constant $0<A_p<\infty$ depending only on p such that for every sequence of independent mean 0 random elements taking values in $\mathcal{X}$,*

$$E\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k}V_i\right\|\right)^p\le A_p\sum_{i=1}^{n}E||V_i||^p, n\ge 1.$$

3. The main result

With the lemmas in hand, the main result may be stated and proved.

Theorem 3.1. *Let $\{V_{n,k}, 1\le k\le k_n, n\ge 1\}$ be an array of rowwise independent random elements taking values in a real separable Rademacher type p $(1\le p\le 2)$ Banach space and let $\{c_n, n\ge 1\}$ be a sequence of positive constants. Suppose for some $J>0$ and some $\delta_1,\delta_2>0$ that*

$$\sum_{n=1}^{\infty}c_n\sum_{k=1}^{k_n}P(||V_{n,k}||>\varepsilon)<\infty \text{ for all } \varepsilon>0, \tag{3.1}$$

$$\sum_{n=1}^{\infty}c_n\left(\sum_{k=1}^{k_n}E\big(||V_{n,k}||^pI(||V_{n,k}||\le\delta_1)\big)\right)^J<\infty, \tag{3.2}$$

and

$$\max_{1\le k\le k_n}\left\|\sum_{i=1}^{k}E(V_{n,i}I(||V_{n,i}||\le\delta_2))\right\|\to 0 \text{ as } n\to\infty. \tag{3.3}$$

Then

$$\sum_{n=1}^{\infty}c_nP\left(\max_{1\le k\le k_n}\left\|\sum_{i=1}^{k}V_{n,i}\right\|>\varepsilon\right)<\infty \text{ for all } \varepsilon>0. \tag{3.4}$$

Proof. Let $\varepsilon>0$ be arbitrary. Choose a positive integer j and a real number A such that

$$2^j>J \text{ and } 0<A<\min\left\{\frac{\varepsilon}{3(3^{j+1}-1)},\delta_1,\delta_2\right\}.$$

For $n\ge 1$ and $1\le k\le k_n$, let

$$V_{n,k}^{(1)}=V_{n,k}I(||V_{n,k}||\le A), V_{n,k}^{(2)}=V_{n,k}I(A<||V_{n,k}||\le\delta_2), \text{ and } V_{n,k}^{(3)}=V_{n,k}I(||V_{n,k}||>\delta_2).$$

Then

$$V_{n,k}=V_{n,k}^{(1)}-EV_{n,k}^{(1)}+V_{n,k}^{(2)}-EV_{n,k}^{(2)}+V_{n,k}^{(3)}+EV_{n,k}^{(1)}+EV_{n,k}^{(2)}, 1\le k\le k_n, n\ge 1$$

and so

$$\sum_{n=1}^{\infty} c_n P\left(\max_{1\le k\le k_n}\left\|\sum_{i=1}^{k} V_{n,i}\right\| > \varepsilon\right)$$
$$\le \sum_{n=1}^{\infty} c_n P\left(\max_{1\le k\le k_n}\left\|\sum_{i=1}^{k}(V_{n,i}^{(1)} - EV_{n,i}^{(1)})\right\| > \frac{\varepsilon}{3}\right)$$
$$+ \sum_{n=1}^{\infty} c_n P\left(\max_{1\le k\le k_n}\left\|\sum_{i=1}^{k}(V_{n,i}^{(2)} - EV_{n,i}^{(2)})\right\| > \frac{\varepsilon}{3}\right)$$
$$+ \sum_{n=1}^{\infty} c_n P\left(\max_{1\le k\le k_n}\left\|\sum_{i=1}^{k} V_{n,i}^{(3)} + \sum_{i=1}^{k}(EV_{n,i}^{(1)} + EV_{n,i}^{(2)})\right\| > \frac{\varepsilon}{3}\right)$$
$$\equiv I + II + III \text{ (say)}.$$

The conclusion (3.4) will be established provided we can show that $I < \infty, II < \infty$, and $III < \infty$.

We first show that $I < \infty$. Note that for all $n \ge 1$,

$$\max_{1\le k\le k_n} ||V_{n,k}^{(1)} - EV_{n,k}^{(1)}|| \le 2A < \frac{2\varepsilon}{3(3^{j+1}-1)} \text{ almost surely (a.s.)}. \tag{3.5}$$

Then

$$P\left(\max_{1\le k\le k_n}\left\|\sum_{i=1}^{k}(V_{n,i}^{(1)} - EV_{n,i}^{(1)})\right\| > \frac{\varepsilon}{3}\right)$$
$$= P\left(\max_{1\le k\le k_n}\left\|\sum_{i=1}^{k}(V_{n,i}^{(1)} - EV_{n,i}^{(1)})\right\| > (3^{j+1}-1)\frac{\varepsilon}{3(3^{j+1}-1)}\right)$$
$$\le C_j P\left(\max_{1\le k\le k_n} ||V_{n,k}^{(1)} - EV_{n,k}^{(1)}|| > \frac{2\varepsilon}{3(3^{j+1}-1)}\right)$$
$$+ \left(P\left(\max_{1\le k\le k_n}\left\|\sum_{i=1}^{k}(V_{n,i}^{(1)} - EV_{n,i}^{(1)})\right\| > \frac{2\varepsilon}{3(3^{j+1}-1)}\right)\right)^{2^j} \text{(by Lemma 2.2)}$$
$$\le 0 + \left(P\left(\max_{1\le k\le k_n}\left\|\sum_{i=1}^{k}(V_{n,i}^{(1)} - EV_{n,i}^{(1)})\right\| > \frac{2\varepsilon}{3(3^{j+1}-1)}\right)\right)^{J} \text{(by (3.5))}$$
$$\le \left(\frac{3(3^{j+1}-1)}{2\varepsilon}\right)^{pJ}\left(E\left(\max_{1\le k\le k_n}\left\|\sum_{i=1}^{k}(V_{n,i}^{(1)} - EV_{n,i}^{(1)})\right\|\right)^{p}\right)^{J} \text{(by the Markov in equality)}$$
$$\le \left(\frac{3(3^{j+1}-1)}{2\varepsilon}\right)^{pJ} A_p^J\left(\sum_{k=1}^{k_n} E||V_{n,k}^{(1)} - EV_{n,k}^{(1)}||^p\right)^{J} \text{(by Lemma 2.3)}$$
$$\le \left(\frac{3(3^{j+1}-1)}{2\varepsilon}\right)^{pJ} 2^{pJ} A_p^J\left(\sum_{k=1}^{k_n} E||V_{n,k}^{(1)}||^p\right)^{J} \text{(by the } c_r-\text{inequality and Jensen's inequality)}$$
$$\le \left(\frac{3(3^{j+1}-1)}{2\varepsilon}\right)^{pJ} 2^{pJ} A_p^J\left(\sum_{k=1}^{k_n} E(||V_{n,k}||^p I(||V_{n,k}|| \le \delta_1))\right)^{J}.$$

Hence, $I < \infty$ by the assumption (3.2).

Next, we show that $II < \infty$. Note that for all $n \geq 1$,

$$P\left(\max_{1\leq k\leq k_n}\left|\left|\sum_{i=1}^{k}(V_{n,i}^{(2)} - EV_{n,i}^{(2)})\right|\right| > \frac{\varepsilon}{3}\right)$$

$$\leq \frac{3}{\varepsilon}E\left(\max_{1\leq k\leq k_n}\left|\left|\sum_{i=1}^{k}(V_{n,i}^{(2)} - EV_{n,i}^{(2)})\right|\right|\right) \text{ (by the Markov inequality)}$$

$$\leq \frac{3}{\varepsilon}A_1\sum_{i=1}^{k_n}E||V_{n,i}^{(2)} - EV_{n,i}^{(2)}|| \text{ (by Lemma 2.3 with } p = 1)$$

$$\leq \frac{6}{\varepsilon}A_1\sum_{i=1}^{k_n}E||V_{n,i}^{(2)}||$$

$$= \frac{6}{\varepsilon}A_1\sum_{i=1}^{k_n}E(||V_{n,i}||I(A < ||V_{n,i}|| \leq \delta_2))$$

$$\leq \frac{6\delta_2}{\varepsilon}A_1\sum_{i=1}^{k_n}E(I(A < ||V_{n,i}|| \leq \delta_2))$$

$$\leq \frac{6\delta_2}{\varepsilon}A_1\sum_{i=1}^{k_n}P(||V_{n,i}|| > A).$$

Hence, $II < \infty$ by the assumption (3.1).

Finally, we show that $III < \infty$. By the assumption (3.3),

$$\max_{1\leq k\leq k_n}\left|\left|\sum_{i=1}^{k}(EV_{n,i}^{(1)} + EV_{n,i}^{(2)})\right|\right| = \max_{1\leq k\leq k_n}\left|\left|\sum_{i=1}^{k}E(V_{n,i}I(||V_{n,i}|| \leq \delta_2))\right|\right| \to 0 \text{ as } n \to \infty$$

and so there exists a positive integer N such that

$$\max_{1\leq k\leq k_n}\left|\left|\sum_{i=1}^{k}(EV_{n,i}^{(1)} + EV_{n,i}^{(2)})\right|\right| \leq \frac{\varepsilon}{6} \text{ for all } n \geq N. \tag{3.6}$$

Then for all $n \geq N$,

$$P\left(\max_{1\leq k\leq k_n}\left|\left|\sum_{i=1}^{k}V_{n,i}^{(3)} + \sum_{i=1}^{k}(EV_{n,i}^{(1)} + EV_{n,i}^{(2)})\right|\right| > \frac{\varepsilon}{3}\right)$$

$$\leq P\left(\max_{1\leq k\leq k_n}\left|\left|\sum_{i=1}^{k}V_{n,i}^{(3)}\right|\right| + \max_{1\leq k\leq k_n}\left|\left|\sum_{i=1}^{k}(EV_{n,i}^{(1)} + EV_{n,i}^{(2)})\right|\right| > \frac{\varepsilon}{3}\right)$$

$$\leq P\left(\max_{1\leq k\leq k_n}\left|\left|\sum_{i=1}^{k}V_{n,i}^{(3)}\right|\right| > \frac{\varepsilon}{6}\right) + 0 \text{ (by (3.6))}$$

$$\leq P\left(\bigcup_{k=1}^{k_n}[||V_{n,k}|| > \delta_2]\right)$$

$$\leq \sum_{k=1}^{k_n}P(||V_{n,k}|| > \delta_2).$$

Hence, $III < \infty$ by the assumption (3.1). This completes the proof. □

Remark 3.1.

(i) If the hypotheses of Theorem 3.1 are satisfied with $c_n = 1, n \geq 1,$ then

$$\max_{1 \leq k \leq k_n} \left\| \sum_{i=1}^{k} V_{n,i} \right\| \text{ converges completely to } 0.$$

Thus, when $c_n = 1, n \geq 1,$ Theorem 3.1 improves Theorem 2 of Hernández, Urmeneta, and Volodin [6] from which it can only be concluded that

$$\left\| \sum_{i=1}^{k_n} V_{n,i} \right\| \text{ converges completely to } 0.$$

(ii) Suppose that the hypotheses of Theorem 3.1 are satisfied with $J=1$. Then (3.2) holds with δ_1 replaced with δ_2. In other words, when $J=1$, Theorem 3.1 is equivalent to the same theorem but with a common value of $\delta > 0$ taken for δ_1 and δ_2 in (3.2) and (3.3), respectively. (The argument is identical to that in Remark 3.1 of Hu, Rosalsky, and Volodin [1]).

(iii) In Example 5.1, it is shown that Theorem 3.1 can fail if $1 < p \leq 2$ and the underlying Banach space is not of Rademacher type p.

4. Corollaries

The first corollary follows immediately from Theorem 3.1 and it is originally due to Pingyan and Shixin [7], Theorem 2.1. Hence Theorem 3.1 is a *bona fide* extension of Theorem 2.1 of Pingyan and Shixin [7] from the (real-valued) random variables case to the case of random elements taking values in a real separable Rademacher type p $(1 \leq p \leq 2)$ Banach space.

Corollary 4.1. (Pingyan and Shixin [7], Theorem 2.1). *Let $\{V_{n,k}, 1 \leq k \leq k_n, n \geq 1\}$ be an array of rowwise independent (real-valued) random variables and let $\{c_n, n \geq 1\}$ be a sequence of positive constants. Suppose for some $J \geq 2$ and some $\delta > 0$*

$$\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P(|V_{n,k}| > \varepsilon) < \infty \text{ for all } \varepsilon > 0,$$

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} E\left(V_{n,k}^2 I(|V_{n,k}| \leq \delta) \right) \right)^J < \infty,$$

and

$$\max_{1 \leq k \leq k_n} \left| \sum_{i=1}^{k} E(V_{n,i} I(|V_{n,i}| \leq \delta)) \right| \to 0 \text{ as } n \to \infty.$$

Then

$$\sum_{n=1}^{\infty} c_n P\left(\max_{1\le k\le k_n}\left|\sum_{i=1}^{k} V_{n,i}\right| > \varepsilon\right) < \infty \text{ for all } \varepsilon > 0.$$

Remark 4.1.

(i) In contradistinction to conditions (3.2) and (3.3) of Theorem 3.1 wherein δ_1 is not necessarily equal to δ_2, two of the conditions in Corollary 4.1 involve the same δ.

(ii) In Example 5.2, (a) the conditions of Theorem 3.1 are satisfied with $0 < \delta_1 < \delta_2$, (b) the conditions of Theorem 3.1 are not satisfied if $\delta_1 = \delta_2 > 0$, and (c) the conditions of Corollary 4.1 are not satisfied.

Corollary 4.2. *Let $\{V_{n,k}, 1 \le k \le k_n, n \ge 1\}$ be an array of rowwise independent random elements taking values in a real separable Rademacher type p $(1 \le p \le 2)$ Banach space and let $\{c_n, n \ge 1\}$ be a sequence of positive constants. Let $Q : [0,\infty) \to [0,\infty)$ be such that for some $\delta_1 > 0$ and some $0 < C < \infty$,*

$$x^p \le CQ(x) \text{ for all } x \in (0, \delta_1]. \tag{4.1}$$

Suppose that (3.1) holds, (3.3) holds for some $\delta_2 > 0$, and

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} EQ(||V_{n,k}||)\right)^J < \infty \text{ for some } J > 0. \tag{4.2}$$

Then (3.4) holds.

Proof. In view of Theorem 3.1, it suffices to verify (3.2) with δ_1 as in (4.1) and J as in (4.2). To this end,

$$\begin{aligned}
&\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} E(||V_{n,k}||^p I(||V_{n,k}|| \le \delta_1))\right)^J \\
&= \sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} E\left(\frac{||V_{n,k}||^p}{Q(||V_{n,k}||)} I(0 < ||V_{n,k}|| \le \delta_1) Q(||V_{n,k}||)\right)\right)^J \\
&\le C\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} EQ(||V_{n,k}||)\right)^J \text{ (by (4.1))} \\
&< \infty \text{ (by (4.2))}
\end{aligned}$$

thereby verifying (3.2). □

The next corollary follows immediately from Corollary 4.2 since (4.1) is immediate with $C = \delta_1 = 1$ and $Q(x) = x^q, x \ge 0$.

Corollary 4.3. *Let $\{V_{n,k}, 1 \le k \le k_n, n \ge 1\}$ be an array of rowwise independent random elements taking values in a real separable Rademacher type p $(1 \le p \le 2)$ Banach space* *and let* $\{c_n, n \geq 1\}$ *be a sequence of positive constants. Suppose that (3.1) holds, (3.3) holds for some* $\delta_2 > 0$*, and*

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} E||V_{n,k}||^q \right)^J < \infty \text{ for some } 0 < q \leq p \text{ and some } J > 0. \tag{4.3}$$

Then (3.4) holds.

Corollary 4.4. *Let* $\{V_{n,k}, 1 \leq k \leq k_n, n \geq 1\}$ *be an array of rowwise independent mean 0 random elements taking values in a real separable Rademacher type* p $(1 \leq p \leq 2)$ *Banach space and let* $\{c_n, n \geq 1\}$ *be a sequence of positive constants. Suppose that (3.1) and (4.3) hold and*

$$\sum_{k=1}^{k_n} E(||V_{n,k}||I(||V_{n,k}|| > M)) \to 0 \text{ as } n \to \infty \text{ for some } M > 0. \tag{4.4}$$

Then (3.4) holds.

Proof. In view of Corollary 4.3, it suffices to verify (3.3) with $\delta_2 = M$. To this end,

$$\begin{aligned}
&\max_{1 \leq k \leq k_n} \left|\left| \sum_{i=1}^{k} E(V_{n,i} I(||V_{n,i}|| \leq M)) \right|\right| \\
&\leq \sum_{i=1}^{k_n} ||E(V_{n,i} I(||V_{n,i}|| \leq M))|| \\
&= \sum_{i=1}^{k_n} ||E(V_{n,i} I(||V_{n,i}|| > M))|| \text{ (since the } V_{n,i} \text{ all have mean 0)} \\
&\leq \sum_{i=1}^{k_n} E(||V_{n,i}||I(||V_{n,i}|| > M)) \to 0 \text{ as } n \to \infty
\end{aligned}$$

by (4.4) thereby verifying (3.3) with $\delta_2 = M$. □

Remark 4.2. It is easy to see that a sufficient condition for (4.4) is that

$$\sum_{k=1}^{k_n} E||V_{n,k}||^r \to 0 \text{ as } n \to \infty \text{ for some } r \geq 1. \tag{4.5}$$

Moreover, if (4.3) holds with $q \geq 1$ and $\liminf_{n \to \infty} c_n > 0$, then (4.5) holds with $r = q$.

An array of random elements $\{V_{n,k}, 1 \leq k \leq k_n, n \geq 1\}$ is said to be *stochastically dominated* by a random variable X if there exists a constant $D < \infty$ such that

$$P(||V_{n,k}|| > x) \leq DP(D|X| > x) \tag{4.6}$$

for all $x \geq 0$ and all $1 \leq k \leq k_n$ and all $n \geq 1$.

Let $\{a_{n,k}, 1 \leq k \leq k_n, n \geq 1\}$ be an array of constants (called *weights*). The following corollary establishes a complete convergence result for weighted sums. In the proof of Corollary 4.5, the symbol C denotes a generic constant $(0 < C < \infty)$ which is not necessary the same one in each appearance.

Corollary 4.5. *Let $\{V_{n,k}, 1 \le k \le k_n, n \ge 1\}$ be an array of rowwise independent random elements taking values in a real separable Rademacher type p $(1 \le p \le 2)$ Banach space. Suppose that $\{V_{n,k}, 1 \le k \le k_n, n \ge 1\}$ is stochastically dominated by a random variable X. Let $\{a_{n,k}, 1 \le k \le k_n, n \ge 1\}$ be an array of constants such that*

$$\sum_{k=1}^{k_n} |a_{n,k}|^p = \mathcal{O}(n^{-\alpha}) \text{ for some } \alpha > 0. \tag{4.7}$$

Suppose that

$$k_n = o(n^{\alpha/p}) \tag{4.8}$$

and

$$E|X|^p < \infty. \tag{4.9}$$

Then for all $\beta < \alpha - 1$,

$$\sum_{n=1}^{\infty} n^\beta P\left(\max_{1 \le k \le k_n} \left\| \sum_{i=1}^{k} a_{n,i} V_{n,i} \right\| > \varepsilon\right) < \infty \text{ for all } \varepsilon > 0. \tag{4.10}$$

Proof. We may assume that $\beta \ge -1$ since the corollary is only of interest in this case. Let $D < \infty$ be as in (4.6). We only need to verify with $c_n = n^\beta, n \ge 1$ that the assumptions (3.1), (3.3), and (4.3) with $q = p$ of Corollary 4.3 hold with $a_{n,k}V_{n,k}$ playing the role of $V_{n,k}$ in the formulation of that corollary.

To verify (3.1), note that for $\beta < \alpha - 1$ and $\varepsilon > 0$,

$$\begin{aligned}
\sum_{n=1}^{\infty} n^\beta \sum_{k=1}^{k_n} P(\|a_{n,k}V_{n,k}\| > \varepsilon) &\le D \sum_{n=1}^{\infty} n^\beta \sum_{k=1}^{k_n} P\left(|X| > \frac{\varepsilon}{D|a_{n,k}|}\right) \text{ (by stochastic domination)} \\
&\le D \sum_{n=1}^{\infty} n^\beta \sum_{k=1}^{k_n} D^p \varepsilon^{-p} |a_{n,k}|^p E|X|^p \text{ (by the Markov inequality)} \\
&\le C \sum_{n=1}^{\infty} n^\beta \sum_{k=1}^{k_n} |a_{n,k}|^p \text{ (by (4.9))} \\
&\le C \sum_{n=1}^{\infty} n^{\beta - \alpha} \text{ (by (4.7))} \\
&< \infty \text{ (since } \beta < \alpha - 1)
\end{aligned}$$

thereby verifying (3.1).

To verify (3.3), note that for any $\delta_2 > 0$,

$$\begin{aligned}
&\max_{1\le k\le k_n}\left|\left|\sum_{i=1}^{k}E(a_{n,i}V_{n,i}I(||a_{n,i}V_{n,i}|| \le \delta_2))\right|\right| \\
&\le \sum_{i=1}^{k_n}|a_{n,i}|E||V_{n,i}|| \\
&\le D^2E|X|k_n \max_{1\le i\le k_n}|a_{n,i}| \text{ (by stochastic domination)} \\
&\le C\frac{k_n}{n^{\alpha/p}} \text{ (by (4.9) and (4.7))} \\
&\to 0 \text{ as } n\to\infty \text{ (by (4.8))}
\end{aligned}$$

thereby verifying (3.3).

Finally, to verify (4.3) with $q = p$, note that for any $J > \frac{\beta+1}{\alpha} \ge 0$,

$$\begin{aligned}
&\sum_{n=1}^{\infty}n^{\beta}\left(\sum_{k=1}^{k_n}E||a_{n,k}V_{n,k}||^p\right)^J \\
&= \sum_{n=1}^{\infty}n^{\beta}\left(\sum_{k=1}^{k_n}|a_{n,k}|^pE||V_{n,k}||^p\right)^J \\
&\le \sum_{n=1}^{\infty}n^{\beta}\left(\sum_{k=1}^{k_n}|a_{n,k}|^pD^{p+1}E|X|^p\right)^J \text{ (by stochastic domination)} \\
&\le C\sum_{n=1}^{\infty}n^{\beta-\alpha J} \text{ (by (4.9) and (4.7))} \\
&< \infty \left(\text{since } J > \frac{\beta+1}{\alpha}\right)
\end{aligned}$$

thereby verifying (4.3). □

Remark 4.3. There is a tradeoff in Corollary 4.5 involving p and the hypotheses; the larger is p, the stronger is the Rademacher type p assumption and conditions (4.8) and (4.9) but the weaker is condition (4.7). Moreover, the larger is α, the stronger is condition (4.7) as well as the conclusion (4.10).

5. Examples

The following modification of an example presented in Kuczmaszewska and Szynal [8] shows that Theorem 3.1 can fail if $1 < p \le 2$ and the underlying Banach space is not of Rademacher type p.

Example 5.1. Let ℓ_1 denote the real separable Banach space of absolutely summable real sequences $x = (x_1, x_2, \ldots)$ with norm $||x|| = \sum_{i=1}^{\infty}|x_i|$ and let $p \in (1, 2]$. It is well known that ℓ_1 is not of Rademacher type p. Let e_k denote the k-th element of the standard basis in ℓ_1; that is, e_k is the element of ℓ_1 having 1 for its k-th coordinate and 0 for the other coordinates. Let $\{R_{n,k}, 1 \le k \le n, n \ge 1\}$ be an array of rowwise independent Rademacher random variables:

$$P(R_{n,k} = 1) = P(R_{n,k} = -1) = \frac{1}{2}, 1 \le k \le n, n \ge 1.$$

Define

$$V_{n,k} = \frac{R_{n,k} e_k}{n}, 1 \le k \le n, n \ge 1.$$

Thus $\{V_{n,k}, 1 \le k \le n, n \ge 1\}$ is an array of rowwise independent random elements in ℓ_1. Let

$$c_n = 1, n \ge 1, \delta_1 = \delta_2 = 1, \text{ and } J > \frac{1}{p-1}.$$

Note that

$$||V_{n,k}|| = \frac{1}{n} \text{a.s.}, 1 \le k \le n, n \ge 1.$$

It is easy to verify that (3.1), (3.2), and (3.3) hold. However,

$$\left\|\sum_{i=1}^{k} V_{n,i}\right\| = \left\|\frac{1}{n}\sum_{i=1}^{k} R_{n,i} e_i\right\| = \frac{1}{n}\sum_{i=1}^{k} 1 = \frac{k}{n} \text{a.s.}, 1 \le k \le n, n \ge 1$$

and so for all ε in (0, 1),

$$\sum_{n=1}^{\infty} P\left(\max_{1\le k\le n}\left\|\sum_{i=1}^{k} V_{n,i}\right\| > \varepsilon\right) = \sum_{n=1}^{\infty} P\left(\left\|\sum_{i=1}^{n} V_{n,i}\right\| > \varepsilon\right) = \sum_{n=1}^{\infty} 1 = \infty.$$

Thus (3.4) fails.

In the following example, the hypotheses of Theorem 3.1 are satisfied but those of Corollary 4.1 are not. Example 5.2 is a modification of Example 4.1 of Hu, Rosalsky, and Volodin [1].

Example 5.2. Let $p = 2, J > 1,$ and define sequences $\{p_n, n \ge 1\}, \{k_n, n \ge 1\},$ and $\{c_n, n \ge 1\}$ as follows. For $n \ge 1$, let

$$p_n = \frac{1}{2^n}, k_n = \begin{cases} \lfloor 2^n n(\log(n+1))^2 \rfloor, & n \text{ odd} \\ \left\lfloor \dfrac{2^n}{n(\log(n+1))^2} \right\rfloor, & n \text{ even} \end{cases}, \text{ and } c_n = \begin{cases} \dfrac{1}{n^2(\log(n+1))^4}, & n \text{ odd} \\ 1, & n \text{ even.} \end{cases}$$

Let $0 < a < \infty$. Let $\{V_{n,k}, 1 \le k \le k_n, n \ge 1\}$ be an array of rowwise independent random variables where

$$P(V_{n,k} = -a) = p_n, P\left(V_{n,k} = \frac{a p_n}{1 - p_n}\right) = 1 - p_n, 1 \le k \le k_n, n \ge 1.$$

Note that

$$\frac{ap_1}{1-p_1} = a \text{ and } \frac{ap_n}{1-p_n} \downarrow 0. \tag{5.1}$$

In Hu, Rosalsky, and Volodin [1], it is shown that

(i) (3.1) holds
and
(ii) (3.2) holds if $0 < \delta_1 < a$ and (3.2) fails if $\delta_1 \geq a$.
Next, if $\delta_2 \geq a$, it follows from (5.1) that

$$\max_{1\leq k\leq k_n}\left|\sum_{i=1}^{k} E(V_{n,i}I(|V_{n,i}| \leq \delta_2))\right| = \max_{1\leq k\leq k_n}\left|\sum_{i=1}^{k} EV_{n,i}\right| = \max_{1\leq k\leq k_n}\left|\sum_{i=1}^{k} 0\right| = 0.$$

On the other hand if $0 < \delta_2 < a$, then it follows from (5.1) that for all large n,

$$\begin{aligned}
&\max_{1\leq k\leq k_n}\left|\sum_{i=1}^{k} E(V_{n,i}I(|V_{n,i}| \leq \delta_2))\right| \\
&\geq \left|\sum_{i=1}^{k_n} E(V_{n,i}I(|V_{n,i}| \leq \delta_2))\right| \\
&= \sum_{i=1}^{k_n} \frac{ap_n}{1-p_n}\cdot(1-p_n) = ak_np_n \not\to 0 \text{ as } n\to\infty
\end{aligned}$$

since $k_np_n \sim n(\log n)^2$ for n odd. Hence

(iii) (3.3) holds if $\delta_2 \geq a$ and (3.3) fails if $0 < \delta_2 < a$.

It follows from (i), (ii), and (iii) that for $p = 2, J > 1,$ and $0 < \delta_1 < a \leq \delta_2$ that the hypotheses of Theorem 3.1 are satisfied and hence (3.4) holds. But in view of (ii) and (iii), the hypotheses of Theorem 3.1 are not satisfied if $\delta_1 = \delta_2 > 0$ and there does not exist a value of $\delta > 0$ satisfying the hypotheses of Corollary 4.1.

The following example shows that Theorem 3.1 can fail if (3.3) is weakened to (1.3); that is, under the conditions of Theorem 1.1, the conclusion (3.4) of Theorem 3.1 does not necessarily hold.

Example 5.3. Define sequences $\{k_n, n \geq 1\}$ and $\{c_n, n \geq 1\}$ as follows. For $n \geq 1$, let

$$k_n = 2n \text{ and } c_n = \frac{1}{n}.$$

Let $\{V_{n,k}, 1 \leq k \leq k_n, n \geq 1\}$ be an array of rowwise independent random variables where for $n \geq 1$,

$$P\left(V_{n,k} = -\frac{1}{n+2}\right) = \frac{n^2+2n}{(n+1)^2} \text{ and } P(V_{n,k} = n) = \frac{1}{(n+1)^2}, 1 \leq k \leq n$$

and

$$P\left(V_{n,k} = \frac{1}{n+2}\right) = \frac{n^2+2n}{(n+1)^2} \text{ and } P(V_{n,k} = -n) = \frac{1}{(n+1)^2}, n+1 \le k \le 2n.$$

We first verify that (3.1) holds. For arbitrary $\varepsilon > 0$ and all large n,

$$\begin{aligned} c_n \sum_{k=1}^{k_n} P(|V_{n,k}| > \varepsilon) &= \frac{1}{n} \sum_{k=1}^{2n} P(|V_{n,k}| = n) \\ &= \frac{1}{n}\left(n \cdot \frac{1}{(n+1)^2} + n \cdot \frac{1}{(n+1)^2}\right) \\ &= \frac{2}{(n+1)^2} \end{aligned}$$

which is summable and so (3.1) holds.

Next, let $p = 2, J > 0$, and $\delta_1 > 0$. We verify that (3.2) holds. Now for all large n,

$$\begin{aligned} & c_n \left(\sum_{k=1}^{k_n} E(|V_{n,k}|^2 I(|V_{n,k}| \le \delta_1)) \right)^J \\ &= \frac{1}{n} \left(\sum_{k=1}^{2n} E(V_{n,k}^2 I(|V_{n,k}| \le \delta_1)) \right)^J \\ &= \frac{1}{n} \left(n \cdot \left(-\frac{1}{n+2}\right)^2 \frac{n^2+2n}{(n+1)^2} + n \cdot \left(\frac{1}{n+2}\right)^2 \frac{n^2+2n}{(n+1)^2} \right)^J \\ &\le \frac{1}{n} \left(\frac{2}{n+2}\right)^J \end{aligned}$$

which is summable since $J > 0$ and so (3.2) holds.

We now verify that (3.3) fails. Note that for all $\delta_2 > 0$ and all large n,

$$\begin{aligned} & \max_{1 \le k \le k_n} \left| \sum_{i=1}^{k} E(V_{n,i} I(|V_{n,i}| \le \delta_2)) \right| \\ &= \max_{1 \le k \le 2n} \left| \sum_{i=1}^{k} E\left(V_{n,i} I\left(|V_{n,i}| = \frac{1}{n+2}\right)\right) \right| \\ &\ge \left| \sum_{i=1}^{n} E\left(V_{n,i} I\left(|V_{n,i}| = \frac{1}{n+2}\right)\right) \right| \\ &= \left| n \cdot \left(-\frac{1}{n+2}\right) \frac{n^2+2n}{(n+1)^2} \right| \to 1 \text{ as } n \to \infty. \end{aligned}$$

Thus (3.3) fails.

Next, we verify that (3.4) fails. Note that for all $n \ge 1$,

$$\left| n \cdot \left(-\frac{1}{n+2}\right) \right| = \frac{n}{n+2} \ge \frac{1}{3} > \frac{1}{4} \tag{5.2}$$

and so

$$\begin{aligned}\sum_{n=1}^{\infty} c_n P\left(\max_{1\le k\le k_n}\left|\sum_{i=1}^{k} V_{n,i}\right| > \frac{1}{4}\right) &= \sum_{n=1}^{\infty}\frac{1}{n}P\left(\max_{1\le k\le 2n}\left|\sum_{i=1}^{k} V_{n,i}\right| > \frac{1}{4}\right)\\ &\ge \sum_{n=1}^{\infty}\frac{1}{n}P\left(\left|\sum_{i=1}^{n} V_{n,i}\right| > \frac{1}{4}\right)\\ &\ge \sum_{n=1}^{\infty}\frac{1}{n}P\left(\left|\sum_{i=1}^{n} V_{n,i}\right| = \left|n\cdot\left(-\frac{1}{n+2}\right)\right|\right)(\text{by }(5.2))\\ &\ge \sum_{n=1}^{\infty}\frac{1}{n}P\left(\bigcap_{i=1}^{n}\left[V_{n,i} = -\frac{1}{n+2}\right]\right)\\ &= \sum_{n=1}^{\infty}\frac{1}{n}\prod_{i=1}^{n}P\left(V_{n,i} = -\frac{1}{n+2}\right)\\ &= \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{n^2+2n}{(n+1)^2}\right)^n\\ &= \sum_{n=1}^{\infty}\frac{1}{n}\left(1-\frac{1}{(n+1)^2}\right)^n\\ &\ge \sum_{n=1}^{\infty}\frac{1}{n}\left(1-\frac{1}{n}\right)^n\\ &= \sum_{n=1}^{\infty}\frac{1}{n}(1+o(1))e^{-1}\\ &= \infty.\end{aligned}$$

Thus (3.4) fails.

However, for any $\delta_2 > 0$ and all large n,

$$\begin{aligned}\sum_{k=1}^{k_n} E(V_{n,k}I(|V_{n,k}|\le\delta_2)) &= \sum_{k=1}^{2n} E\left(V_{n,k}I\left(|V_{n,k}| = \frac{1}{n+2}\right)\right)\\ &= n\cdot\left(-\frac{1}{n+2}\right)\frac{n^2+2n}{(n+1)^2} + n\cdot\left(\frac{1}{n+2}\right)\frac{n^2+2n}{(n+1)^2}\\ &= 0\end{aligned}$$

and so (1.3) holds. Noting that (3.1) and (3.2) coincide with (1.1) and (1.2), respectively, it follows from (1.1), (1.2), (1.3), and Theorem 1.1 that (1.4) holds.

The following example illustrates Corollary 4.3. The example is a modification of Example 5.1 of Hu, Rosalsky, and Wang [3].

Example 5.4. Let $\{V_n, n\ge 1\}$ be a sequence of independent and identically distributed random elements taking values in a real separable Rademacher type p $(1\le p\le 2)$ Banach space. Let $\alpha\ge 0, J>0, 0<q\le p,$ and $\lambda > ((q-1)J-\alpha-1)\vee 0$. Suppose that

$$E||V_1||^{\max\{q,r,1\}} < \infty \text{ for some } r > \frac{(\alpha+2)qJ}{\alpha+J+\lambda+1}.$$

Set

$$V_{n,k} = \frac{V_k}{n^{\frac{\alpha+J+\lambda+1}{qJ}}}, 1 \le k \le n, n \ge 1.$$

Let $k_n = n$ and $c_n = n^\alpha, n \ge 1$. In Example 5.1 of Hu, Rosalsky, and Wang [3], it is shown that (3.1) and (4.3) hold. We now verify that (3.3) holds with $\delta_2 = 1$. Note that

$$\begin{aligned}\max_{1\le k\le n}\left|\left|\sum_{i=1}^{k} E(V_{n,i}I(||V_{n,i}|| \le 1))\right|\right| &\le \sum_{i=1}^{n} E||V_{n,i}|| \\ &= \sum_{i=1}^{n} E\left(\frac{||V_1||}{n^{\frac{\alpha+J+\lambda+1}{qJ}}}\right) \\ &= \frac{E||V_1||}{n^{\frac{\alpha+J+\lambda+1-qJ}{qJ}}} \to 0 \text{ as } n \to \infty\end{aligned}$$

since $\lambda > (q-1)J - \alpha - 1$ and so (3.3) holds. Thus, by Corollary 4.3,

$$\sum_{n=1}^{\infty} n^\alpha P\left(\max_{1\le k\le n}\left|\left|\sum_{i=1}^{k} V_i\right|\right| > \varepsilon n^{\frac{\alpha+J+\lambda+1}{qJ}}\right) < \infty \text{ for all } \varepsilon > 0.$$

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